CS103
Winter 2025

Lecture 10:

Graph Theory

Part 2 of 3

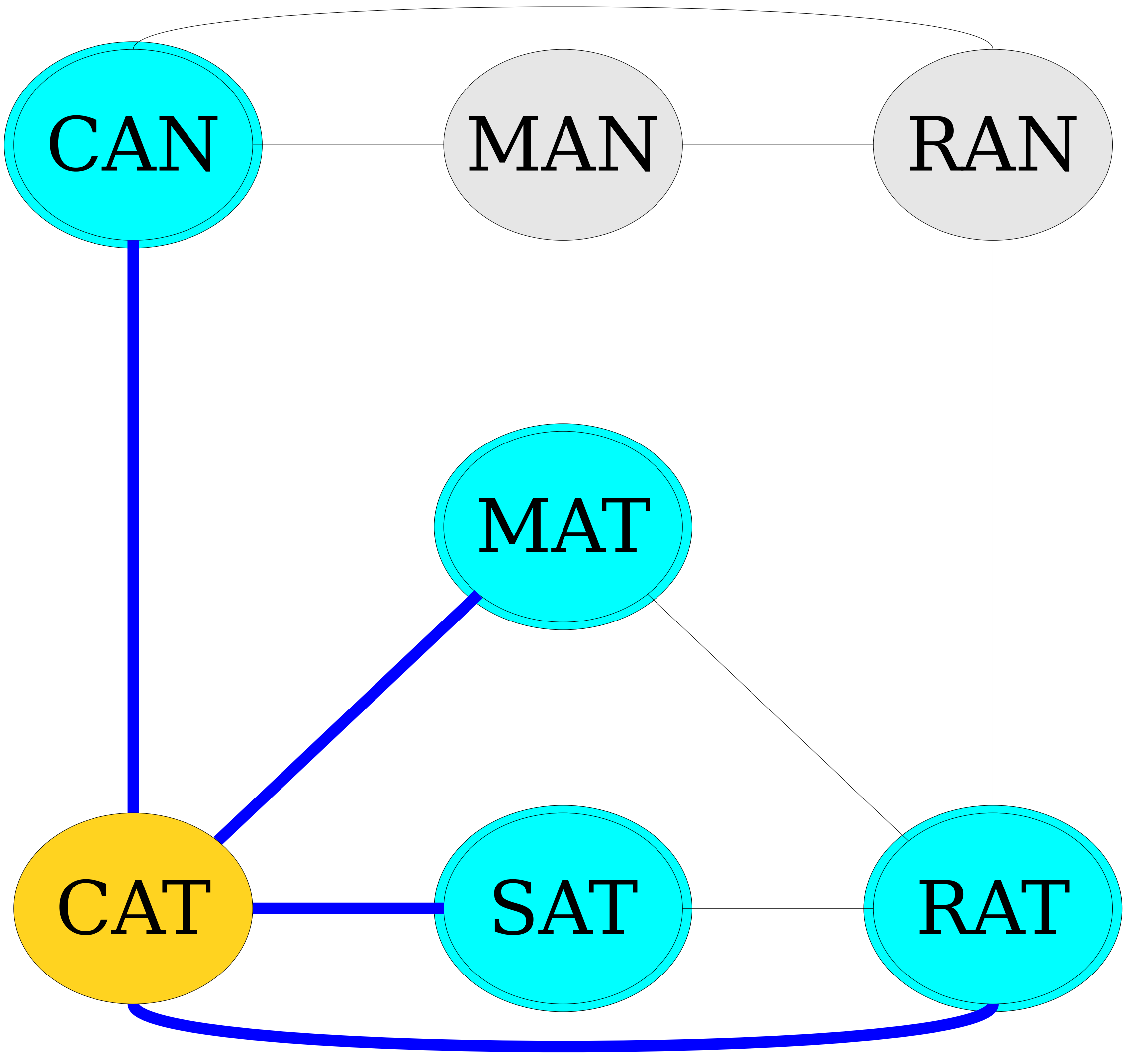
Outline for Today

- ***Walks, Paths, and Reachability***
  - Walking around a graph.
- ***Application: Local Area Networks***
  - Graphs meet computer networking.
- ***Trees***
  - A fundamental class of graphs.

Recap from Last Time

Graphs and Digraphs

- A ***graph*** is a pair $G = (V, E)$ of a set of nodes V and set of edges E.
  - Nodes can be anything.
  - Edges are ***unordered pairs*** of nodes. If $\{u, v\} \in E$, then there's an edge from u to v.
- A ***digraph*** is a pair $G = (V, E)$ of a set of nodes V and set of directed edges E.
  - Each edge is represented as the ordered pair (u, v) indicating an edge from u to v.



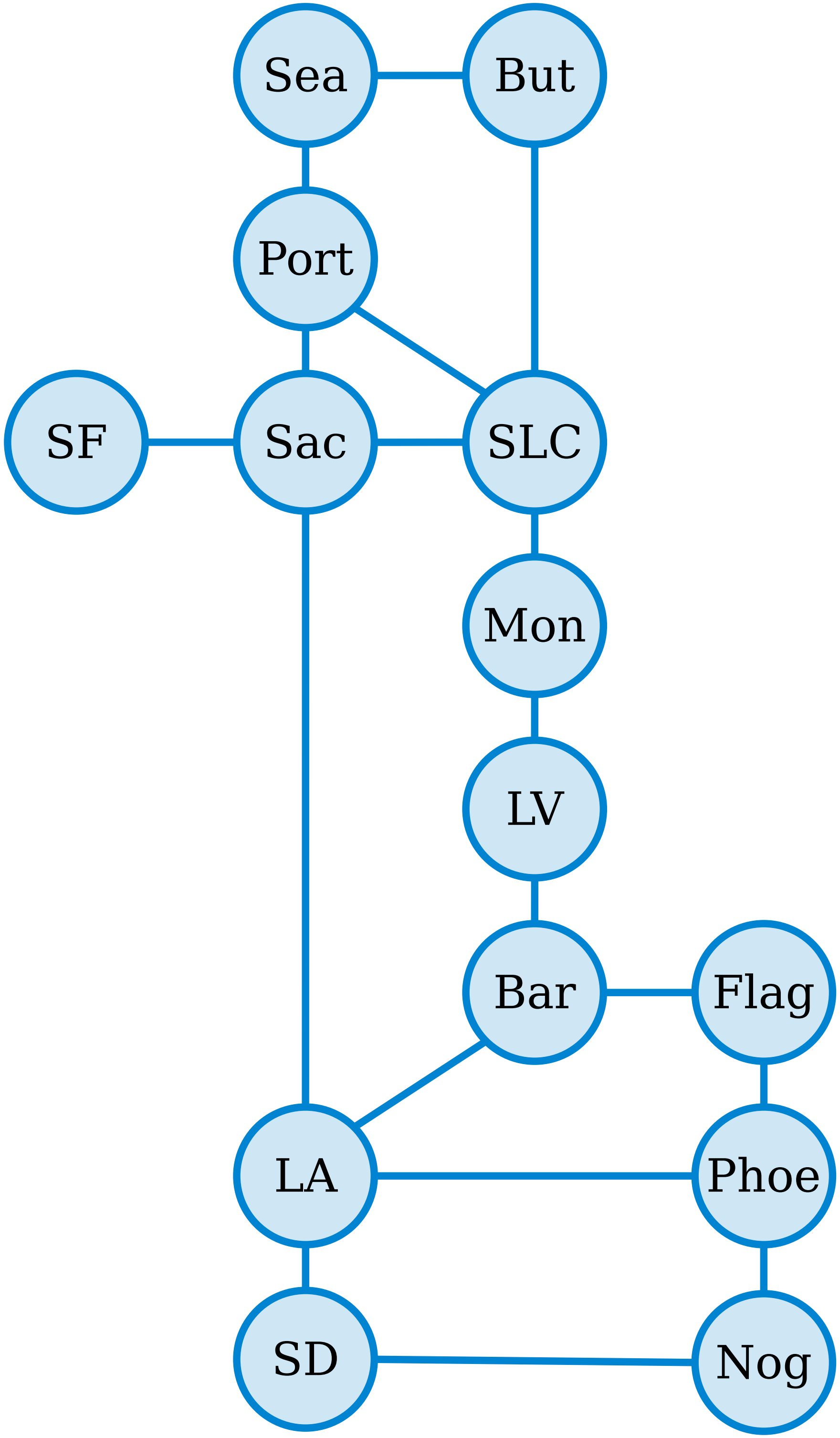
Two nodes in an undirected graph are called ***adjacent*** if there is an edge between them.

Using our Formalisms

- Let $G = (V, E)$ be an (undirected) graph.
- Intuitively, two nodes are adjacent if they're linked by an edge.
- Formally speaking, we say that two nodes $u, v \in V$ are ***adjacent*** if we have $\{u, v\} \in E$.
- There isn't an analogous notion for directed graphs. We usually just say "there's an edge from u to v" as a way of reading $(u, v) \in E$ aloud.

New Stuff!

Walks, Paths, and Reachability



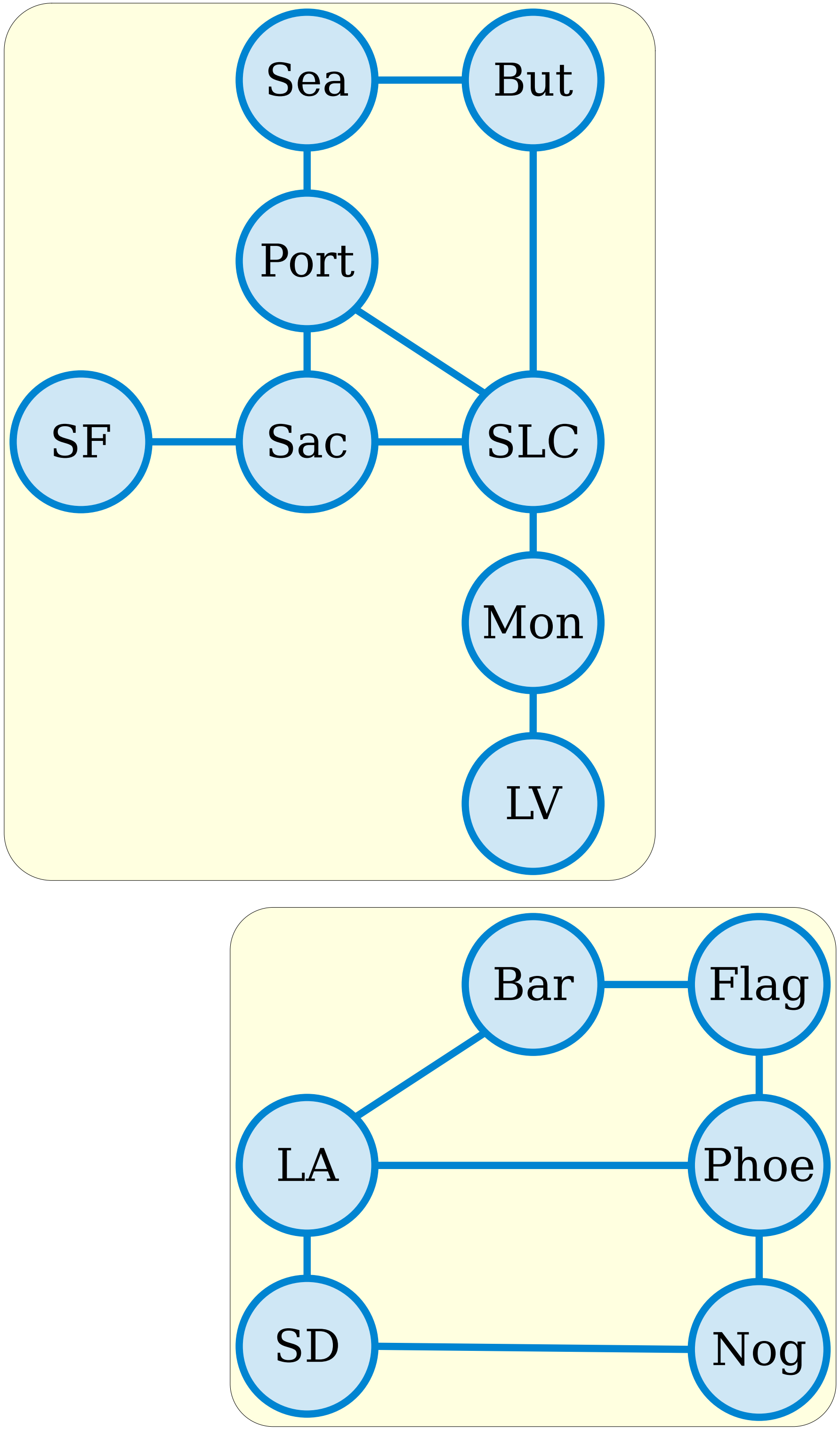
A ***walk*** in a graph $G = (V, E)$ is a sequence of one or more nodes $v_1, v_2, v_3, \ldots, v_n$ such that any two consecutive nodes in the sequence are adjacent.

The ***length*** of the walk $v_1, \ldots, v_n$ is $n - 1$.

A ***closed walk*** in a graph is a walk from a node back to itself. (By convention, a closed walk cannot have length zero.)

A ***path*** in a graph is walk that does not repeat any nodes.

A ***cycle*** in a graph is a closed walk that does not repeat any nodes or edges except the first/last node.



A ***walk*** in a graph $G = (V, E)$ is a sequence of one or more nodes $v_1, v_2, v_3, \ldots, v_n$ such that any two consecutive nodes in the sequence are adjacent.

A ***path*** in a graph is walk that does not repeat any nodes.

A node v is ***reachable*** from a node u when there is a path from u to v.

A graph G is called ***connected*** when all pairs of distinct nodes in G are reachable.

A ***connected component*** (or ***CC***) of G is a set consisting of a node and every node reachable from it.

Fun Facts

- Here's a collection of useful facts about graphs that you can take as a given.
  - ***Theorem:*** If $G = (V, E)$ is a (directed or undirected) graph and $u, v \in V$, then there is a path from u to v if and only if there's a walk from u to v.
  - ***Theorem:*** If G is an undirected graph and C is a cycle in G, then C's length is at least three and C contains at least three nodes.
  - ***Theorem:*** If $G = (V, E)$ is an undirected graph, then every node in V belongs to exactly one connected component of G.
  - ***Theorem:*** If $G = (V, E)$ is a (directed or undirected) graph and $u, y_0, y_1, \ldots, y_m, v$ is a walk from u to v and $v, z_0, z_1, \ldots, z_n, x$ is a walk from v to x, then $u, y_0, y_1, \ldots, y_m, v, z_0, z_1, \ldots, z_n, x$ is a walk from u to x.
- Looking for more practice working with formal definitions? Prove these results!

Time-Out for Announcements!

Things to Have on Your Radar

- Extra credit pre-midterm reflection due Sunday.
- Problem Set 4 releases after class today. Designed to be shorter than usual.
- Make sure to ***review your feedback*** on PS1 and PS2.
  - “Make new mistakes.”
  - Come talk to us if you have questions!
- Exam Tuesday. Check seating assignment and logistics on course website.
- There’s a huge bank of practice problems up on the course website.
- Best of luck – ***you can do this!***

Participation Opt-Out

- By default, all on-campus students have 5% of their grade allocated from lecture attendance and participation.
- If you are an on-campus student and want to opt out, shifting that 5% onto your final exam, fill out the opt-out form on Ed by tonight (Friday) at 11:59 PM.

Back to CS103!

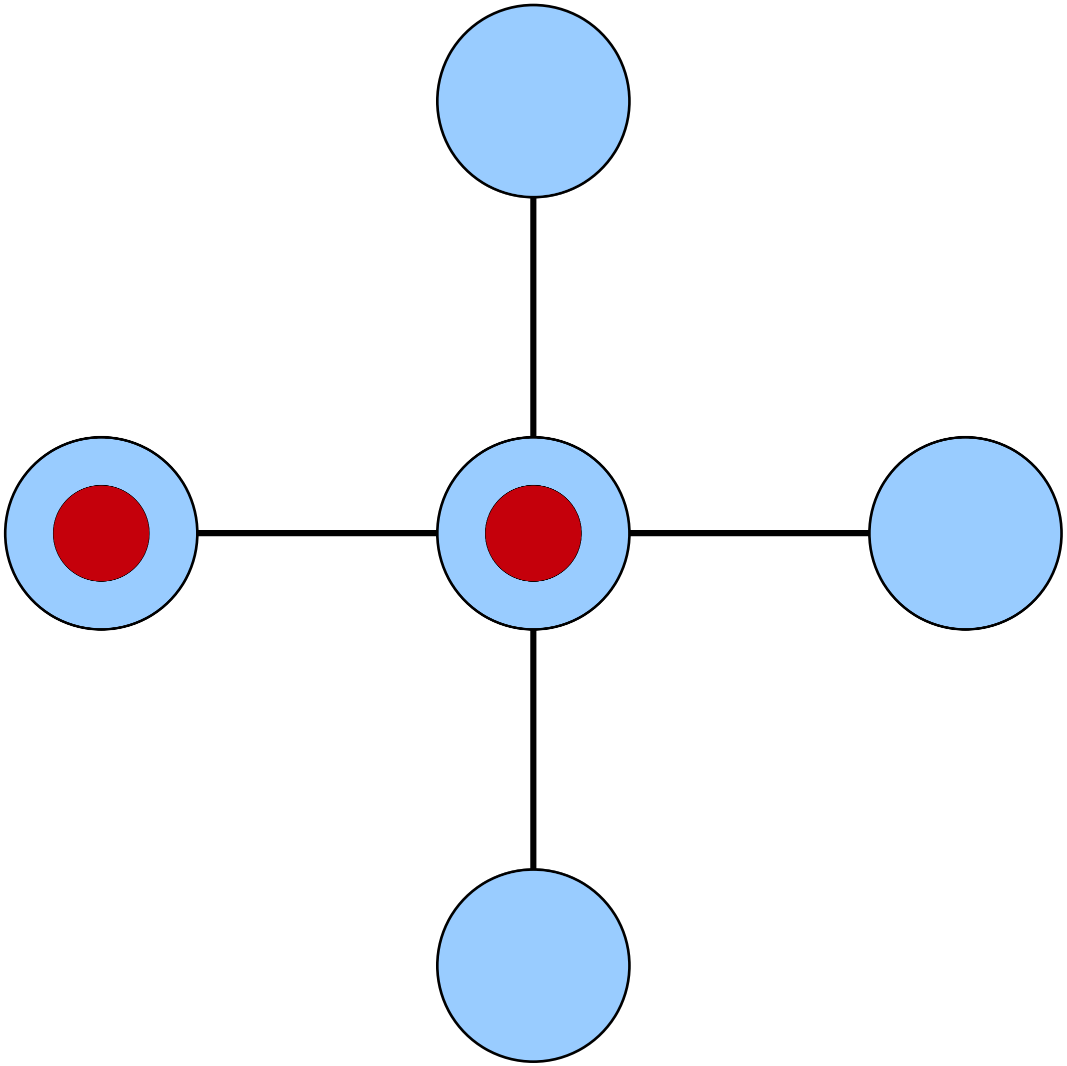
Application: ***Local Area Networks***

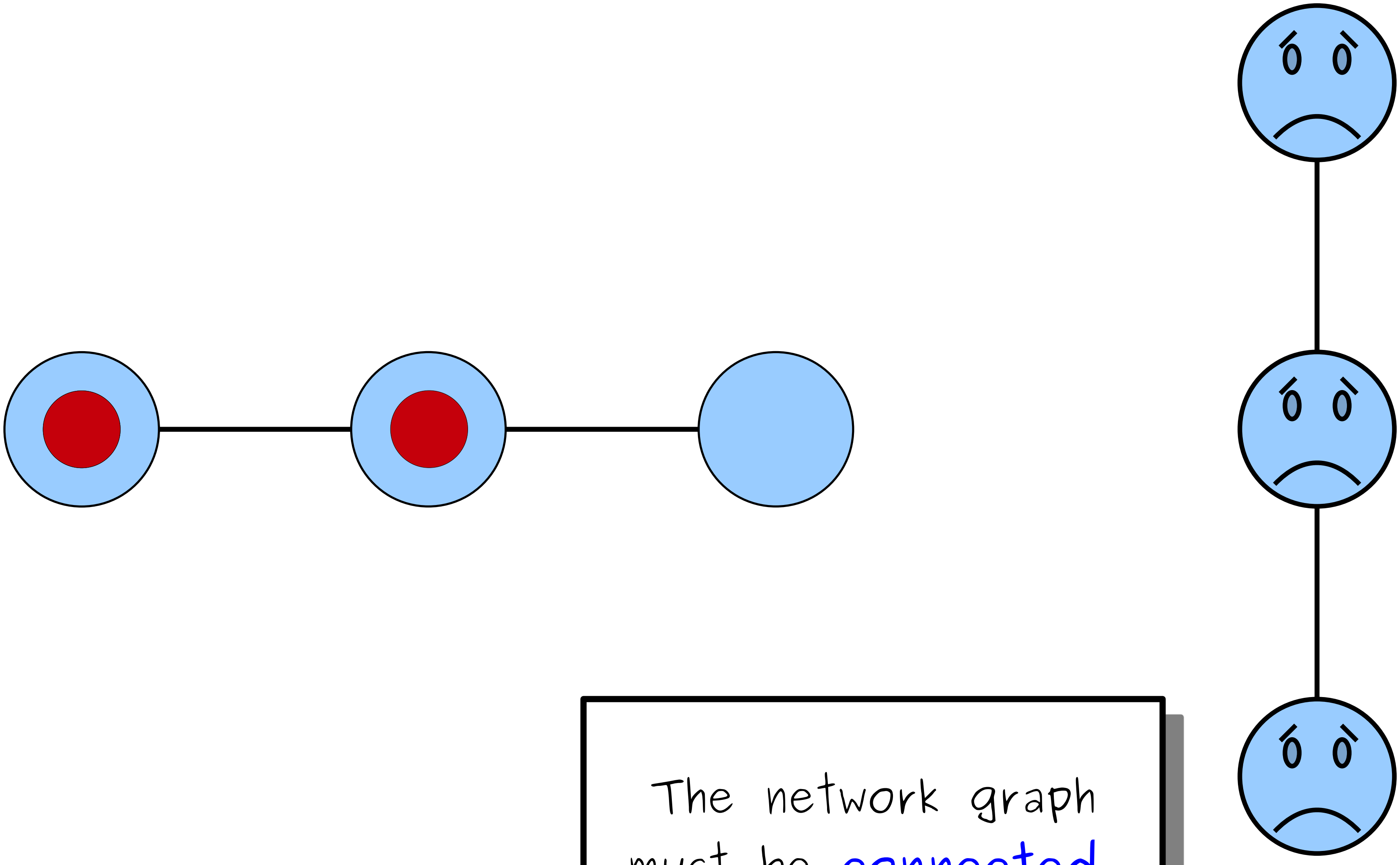
The Internet and LANs

- The internet consists of several separate ***local area networks*** (***LANs***) that are "internetworked" together.
- Local area networks cover small areas – a single hallway in a dorm, an office building, a college campus, etc.
- The internet then links those smaller LANs into one giant network where everyone can talk to everyone.
- ***Focus for today:*** How do messages flow through a LAN?

Message Movement

- When a computer receives a message, it repeats that message on all its links except the one it received the message on.
- The computers don't inspect the message contents or try to be clever – it's purely "came in on link *X*, goes out on all links but *X*."



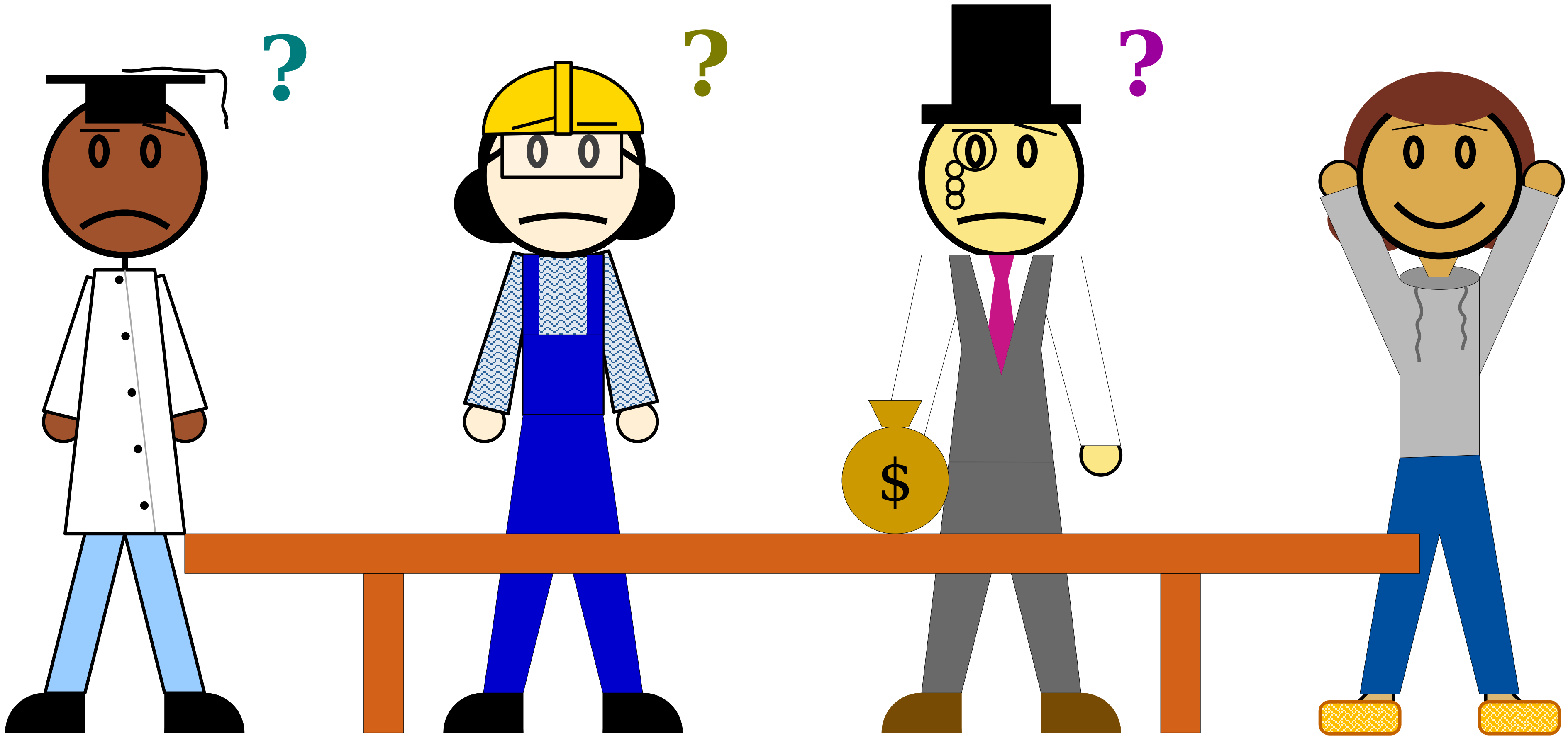


The network graph must be connected.

Broadcast Storms

- A ***broadcast storm*** occurs when there's a cycle in the network graph.
- A single message can repeat forever, or exponentially amplify until the network fails.
- ***Solution:*** Don't let the network graph have any cycles.
- A graph $G = (V, E)$ is called ***acyclic*** if it has no cycles.

You have a collection of computers that need to be wired up into a LAN. How should you choose the shape of the network?



CTO

Connected,
No Cycles

COO

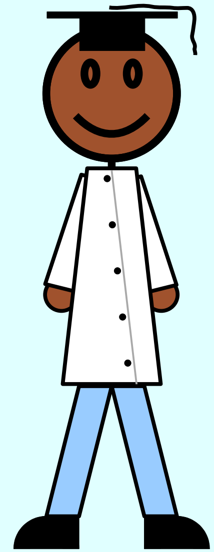
Most Links,
No Cycles

CFO

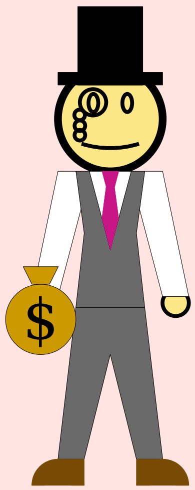
Fewest Links,
Connected

CEO

Do all three!



Connected, Acyclic

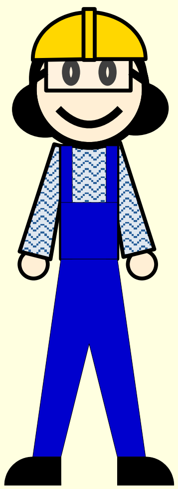


Minimally Connected

(Connected, but deleting any edge disconnects its endpoints.)

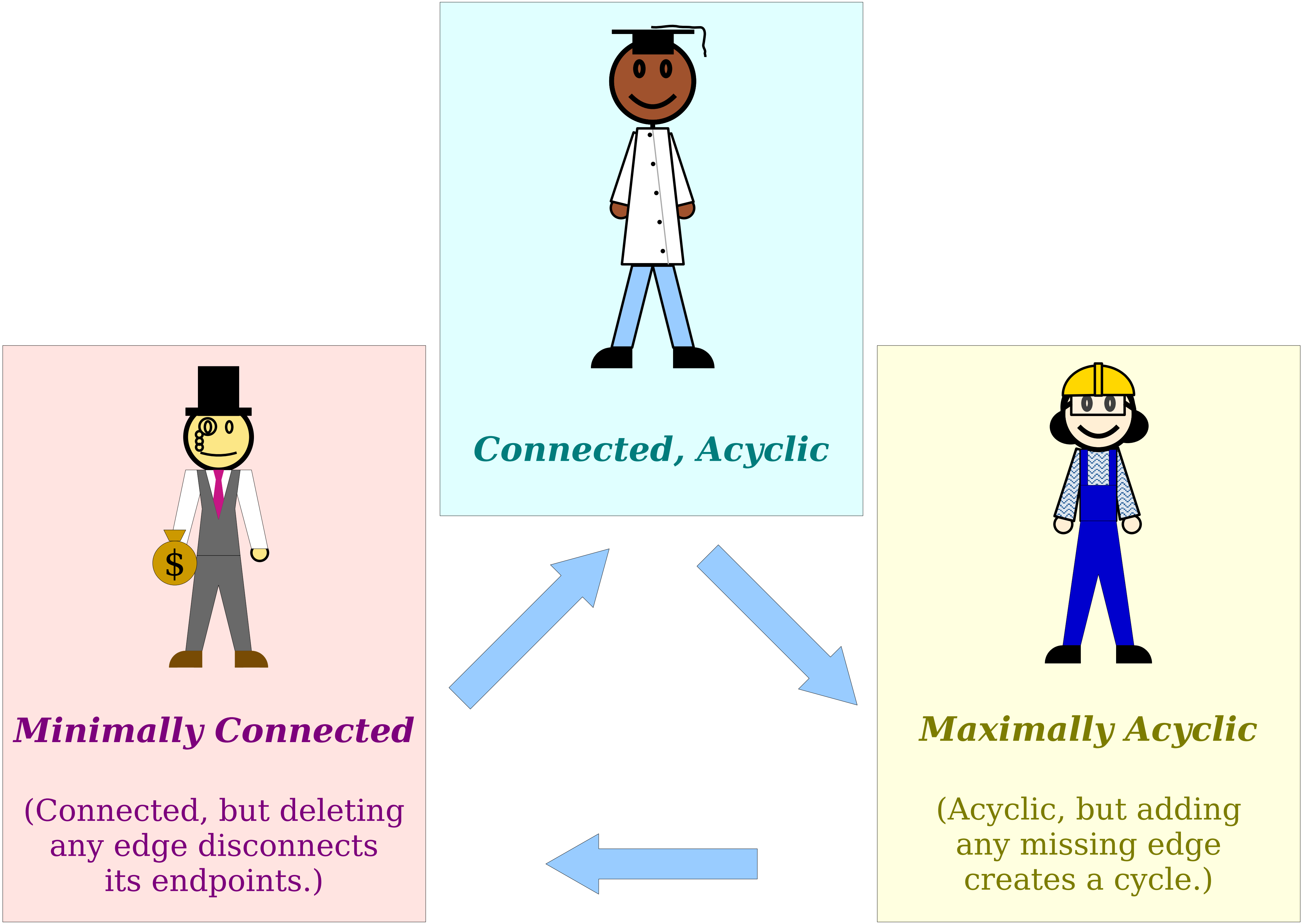
If *any* of these conditions hold, then *all* of these conditions hold.

A graph with any of these properties is called a ***tree***.



Maximally Acyclic

(Acyclic, but adding any missing edge creates a cycle.)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(Connected, but deleting any edge disconnects its endpoints.)
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(Acyclic, but adding any missing edge creates a cycle.)

Theorem: Let $T = (V, E)$ be a graph. If T is connected and acyclic, then T is maximally acyclic.

Proof: Assume T is connected and has no cycles. We need to prove that T is maximally acyclic. We already know that T is acyclic. So choose distinct nodes $x, y \in V$ where $\{x, y\} \notin E$; we'll prove adding $\{x, y\}$ to E closes a cycle.

Because T is connected, there is a path $x, \ldots, y$ from x to y in T. Now add $\{x, y\}$ to E. Then we can form the closed walk $x, \ldots, y, x$. We claim that this is a cycle. To see this, note the following:

- No node is repeated except the start/end node x: nodes $x, \ldots, y$ are all distinct because $x, \ldots, y$ is a path.
- No edge appears twice: none of the edges used in $x, \ldots, y$ are repeated ($x, \ldots, y$ is a path). Furthermore, the edge $\{x, y\}$ isn't repeated since the path $x, \ldots, y$ was formed before $\{x, y\}$ was added to E.

Thus adding $\{x, y\}$ to E closes a cycle, as required. ■

Check the appendix for the
other two steps of the proof.

More to Explore

- A tree kind of seems like a bad way to design a network. (Why?)
- Actual local area networks allow for cycles. They use something called the ***spanning tree protocol*** (***STP***) to selectively disable links to form a tree.
- Routing through the full internet – not just within a LAN – is a fascinating topic in its own right.
- Take CS144 (networking) for details!

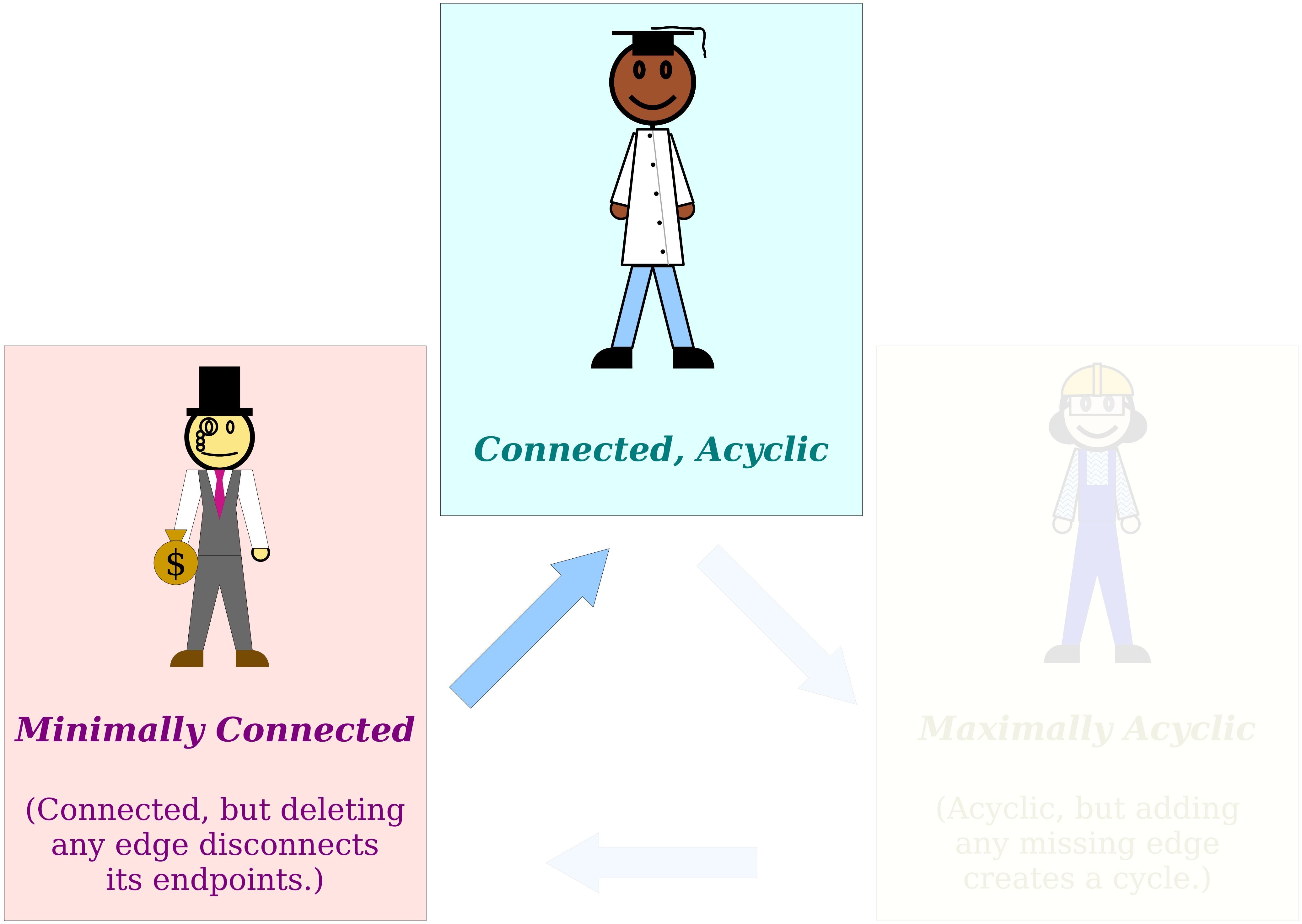
Recap from Today

- ***Walks*** and ***closed walks*** represent ways of moving around a graph. ***Paths*** and ***cycles*** are “redundancy-free” walks and cycles.
- ***Trees*** are graphs that are connected and acyclic. They’re also minimally-connected graphs and maximally-acyclic graphs.
- Trees have applications throughout CS, including networking.

Next Time

- ***The Pigeonhole Principle***
  - A simple, powerful, versatile theorem.
- ***Graph Theory Party Tricks***
  - Applying math to graphs of people!
- ***A Little Movie Puzzle***
  - Who watched what?

Appendix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Minimally Connected
(Connected, but deleting any edge disconnects its endpoints.)
Maximally Acyclic
(Acyclic, but adding any missing edge creates a cycle.)

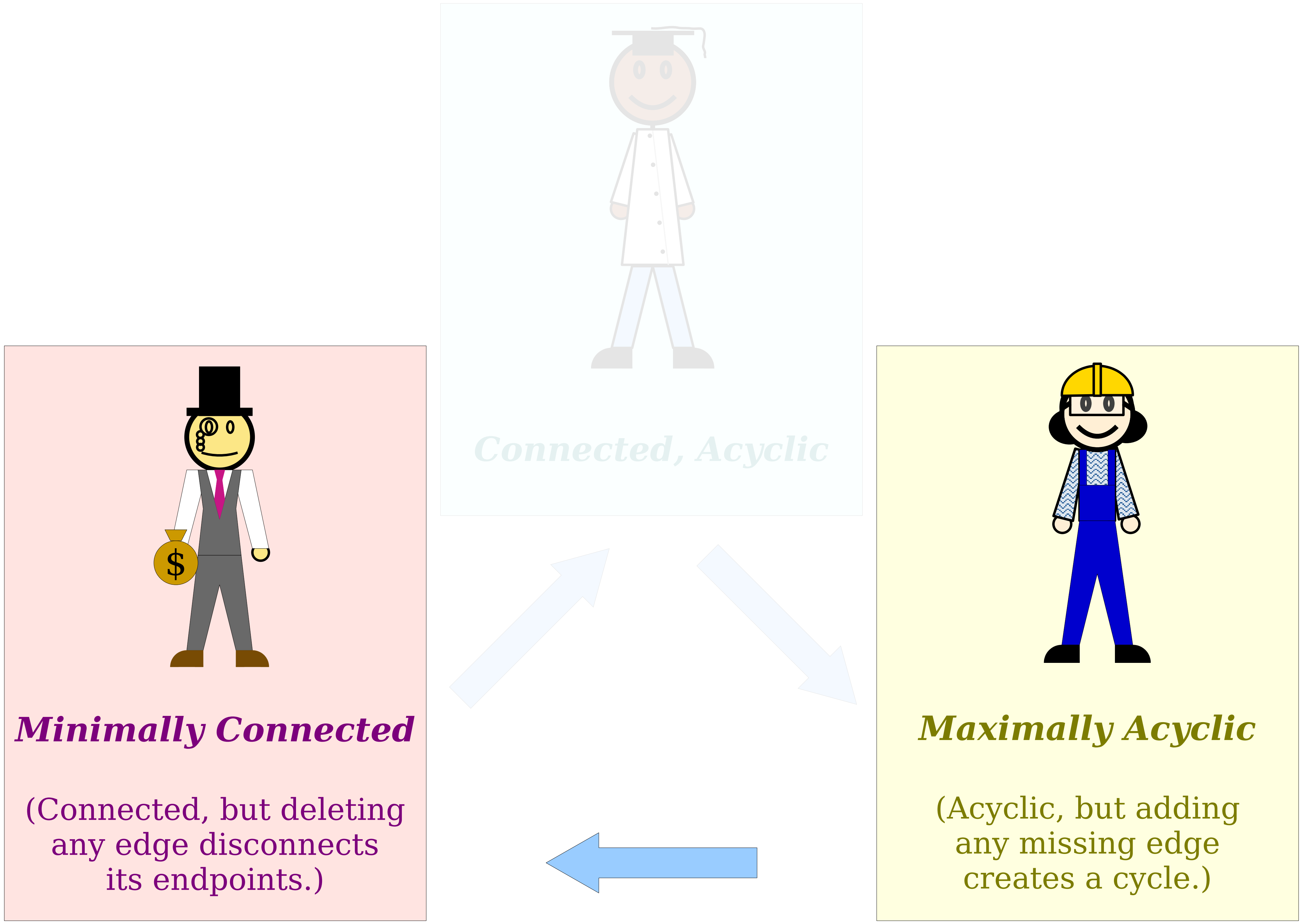
Theorem: Let $T = (V, E)$ be a graph. If T is minimally connected, then T is connected and acyclic.

Proof: Assume T is minimally connected. We need to show that T is connected and acyclic. Since T is minimally connected, it's connected, and so we just need to show that T is acyclic.

Suppose for the sake of contradiction that T contains a cycle $x, \ldots, y, x$. Note in particular that this means $x, \ldots, y$ is a path in T and that this path does not use the edge $\{x, y\}$.

Since T is minimally connected, deleting the edge $\{x, y\}$ from T makes y not reachable from x. However, we said earlier that $x, \ldots, y$ is a path from x to y in T that does not use $\{x, y\}$, so x and y remain reachable after deleting $\{x, y\}$.

We have reached a contradiction, so our assumption was wrong and T is acyclic. ■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Theorem: Let $T = (V, E)$ be a graph. If T is maximally acyclic, then T is minimally connected.

Proof: Assume T is maximally acyclic. We need to prove that T is minimally connected. To do so, we first prove T is connected. Pick any $x, y \in V$ where $x \neq y$; we'll show there's a path from x to y. Consider two cases:

Case 1: $\{x, y\} \in E$. Then x, y is a path from x to y.

Case 2: $\{x, y\} \notin E$. Imagine adding $\{x, y\}$ to E. Since T is maximally acyclic, this closes a cycle $x, \ldots, y, x$ passing through $\{x, y\}$. Then $x, \ldots, y$ is a path in T from x to y.

In either case, we have a path from x to y, as needed.

Next, suppose for the sake of contradiction that there is an edge $\{x, y\} \in E$ where T remains connected after deleting $\{x, y\}$. This means that there is a path $x, \ldots, y$ in T after removing $\{x, y\}$. By adding $\{x, y\}$ to the end of the path, we form a cycle $x, \ldots, y, x$ in T. This is impossible because T is acyclic. We've reached a contradiction, so our assumption was wrong and T is minimally connected. ■